

## ANALYTIC SUBORDINATION FOR BI-FREE CONVOLUTION

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ABSTRACT. In this paper we study some analytic properties of bi-free additive convolution, both scalar and operator-valued. We show that using properties of Voiculescu's subordination functions associated to free additive convolution of operator-valued distributions, simpler formulas for bi-free convolutions can be derived. We use these formulas in order to prove a result about atoms of bi-free additive convolutions.

## 1. INTRODUCTION

In his second paper on bi-free independence [23], Voiculescu provided a linearizing transform for the bi-free additive convolution of compactly supported probability measures in the plane  $\mathbb{R}^2$ . This formula may be re-written to naturally involve the subordination functions of the free additive convolutions of the two faces (marginals) of the two probability measures on  $\mathbb{R}^2$  (see [23, Remark 2.3]). Motivated by the recent work of two of us [12], we note that this connection can be more directly justified by appealing to the operator-valued subordination as introduced in [19], applied to the restriction to upper triangular  $2 \times 2$  complex matrices. While no freeness over any subalgebra of the  $2 \times 2$  matrices appears to be involved in bi-freeness, we find nevertheless a proof (and slight extension) of this formula involving Voiculescu's methods from [19]. Thus, the purpose of this note is to provide a bi-free equivalent of Voiculescu's subordination result [17] and present a few of its most natural consequences (see Propositions 3.4 and 3.5).

The rest of this paper is organized the following way: in Section 2 we provide the necessary background in free and bi-free probability, in Section 3 we prove a subordination relation for bi-free additive convolution of both scalar and operator-valued bi-distributions and for bi-free convolution semigroups. We conclude Section 3 with a few regularity results that follow from this main result. In Section 4 we study analytic properties of the bi-free convolution semigroups introduced in [11]. The last section is dedicated to a discussion of conditional expectations and traciality in the context of bi-freeness.

## 2. NOTATIONS AND BACKGROUND

For the purposes of this note, we only need to consider the definition of the bi-freeness of two pairs of random variables. In that, we follow [23, Section 1.1], and refer to [10, 22] for full details of the analytic and combinatorial aspects of bi-free probability. Consider a noncommutative probability space  $(\mathcal{A}, \varphi)$ . We assume that  $\mathcal{A}$  is endowed with an involution  $*$  and that  $\varphi(x^*) = \overline{\varphi(x)}$  for all  $x \in \mathcal{A}$ . We will always assume that  $\varphi$  is positive and faithful, meaning that

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$\varphi(x^*x) \geq 0$ , with equality if and only if  $x = 0$ . A pair  $((a_1, b_1), (a_2, b_2))$  of two-faced noncommutative random variables in  $(\mathcal{A}, \varphi)$  is said to be bi-free if their distribution satisfies the following property: there are two vector spaces  $\mathcal{X}_1, \mathcal{X}_2$  with distinguished state vectors  $\xi_1, \xi_2$  (i.e.  $\mathcal{X}_j = \mathbb{C}\xi_j \oplus \ker \psi_j$ , with  $\psi_j: \mathcal{X}_j \rightarrow \mathbb{C}$  linear,  $\psi_j(\xi_j) = 1$ ), so that if  $(\mathcal{X}, \ker \psi, \xi) = (\mathcal{X}_1, \ker \psi_1, \xi_1) * (\mathcal{X}_2, \ker \psi_2, \xi_2)$ , and  $\lambda_j, \rho_j$  are the left and right representations of  $\mathcal{L}(\mathcal{X}_j)$  on  $\mathcal{L}(\mathcal{X})$ ,  $j = 1, 2$ , then the joint distribution of  $a_1, a_2, b_1, b_2$  with respect to  $\varphi$  in  $\mathcal{A}$  equals the joint distribution of variables  $\lambda_1(a_1), \lambda_2(a_2), \rho_1(b_1), \rho_2(b_2)$  with respect to  $\varphi_\xi$  in  $\mathcal{L}(\mathcal{X})$ . Here  $\varphi_\xi(T) = \varphi(T(\xi))$ ,  $T \in \mathcal{L}(\mathcal{X})$ . The elements  $a_j$  and  $b_j$  are called the left and right, respectively, faces of  $(a_j, b_j)$ . It follows from the definition that left and right faces of different pairs are classically independent, so that, in particular, they commute.

In [23], Voiculescu shows that if the joint distribution of  $(a_j, b_j)$  is determined by  $\varphi(LR)$ , where  $L$  runs through all monomials in the left face and  $R$  runs through all monomials in the right face,  $j = 1, 2$ , then the same remains true for  $(a_1 + a_2, b_1 + b_2)$ . If in addition  $(\mathcal{A}, \varphi)$  is a  $*$ -probability space in which  $a_j = a_j^*, b_j = b_j^*$ ,  $a_j b_j = b_j a_j$ , then the joint distribution of  $(a_j, b_j)$  coincides with the moments of a compactly supported probability measure  $\eta_j$  in the plane. The correspondence is given via the relation

$$\varphi(a_j^m b_j^n) = \int_{\mathbb{R}^2} t^m s^n d\eta_j(t, s), \quad m, n \in \mathbb{N}, j = 1, 2.$$

Obviously, under this hypothesis, the distribution of  $(a_1 + a_2, b_1 + b_2)$  is itself the joint distribution of two commuting selfadjoint random variables, so that there exists a compactly supported probability measure  $\eta$  on  $\mathbb{R}^2$  whose moments coincide with it. The measure  $\eta$  depends only on  $\eta_1$  and  $\eta_2$  via formulae provided, for example, in [10]. The notation  $\eta = \eta_1 \boxplus \eta_2$  was introduced in [23] and is called the bi-free additive convolution of  $\eta_1$  and  $\eta_2$ . The measure  $\eta$  has the property that its marginals are the free additive convolutions of the marginals of  $\eta_1$  and  $\eta_2$ . More specifically, if  $\mu_j$  is the distribution of  $a_j$  and  $\nu_j$  is the distribution of  $b_j$ , then the first marginal of  $\eta$  is  $\mu_1 \boxplus \mu_2$  and the second marginal of  $\eta$  is  $\nu_1 \boxplus \nu_2$ .

To linearize the bi-free additive convolution, Voiculescu introduced the partial bi-free  $R$ -transform, a function of two complex variables defined on a neighbourhood of zero in  $\mathbb{C}^2$ . We introduce this function, together with its single-variable analogue, and indicate how it allows one to interpret the operation  $\boxplus$  in terms of the single-variable analytic subordination functions [17, 9, 4].

First, define

$$G_{\eta_j}(z, w) = \varphi((z - a_j)^{-1}(w - b_j)^{-1}) = \int_{\mathbb{R}^2} \frac{d\eta_j(t, s)}{(z - t)(w - s)},$$

$$G_{\mu_j}(z) = \varphi((z - a_j)^{-1}) = \int_{\mathbb{R}} \frac{d\mu_j(t)}{z - t}, \quad G_{\nu_j}(w) = \varphi((w - b_j)^{-1}) = \int_{\mathbb{R}} \frac{d\nu_j(t)}{w - t},$$

for  $z \in \mathbb{C} \setminus \sigma(a_j), w \in \mathbb{C} \setminus \sigma(b_j)$ ,  $j = 1, 2$ . Here  $\sigma(T)$  denotes the spectrum of the operator  $T$ . We shall refer to all three of these functions as the Cauchy transforms of the corresponding measures. Observe that they determine uniquely the probability measures in question, and depend only on the distribution of  $(a_j, b_j)$  with respect to  $\varphi$ . Nevertheless, in the following, we will sometimes write  $G_{(a_j, b_j)}$  for  $G_{\eta_j}$ , or  $G_{a_j}$  for  $G_{\mu_j}$  (resp.  $G_{b_j}$  for  $G_{\nu_j}$ ).

It is known from [16] that if one defines  $K_{\mu_j}(z)$  as the inverse of  $G_{\mu_j}(z)$  on a neighbourhood of infinity (so that  $K_{\mu_j}(0) = \infty$ ), then the function  $R_{\mu_j}(z) =$

$K_{\mu_j}(z) - \frac{1}{z}$  is analytic (instead of just meromorphic) on the same neighbourhood of zero and satisfies the relation  $R_{\mu_1 \boxplus \mu_2}(z) = R_{\mu_1}(z) + R_{\mu_2}(z)$  for  $z$  in a small enough neighbourhood of zero. Observe that if we define  $\omega_{a_1}(z) = K_{\mu_1}(G_{\mu_1 \boxplus \mu_2}(z))$  and  $\omega_{a_2}(z) = K_{\mu_2}(G_{\mu_1 \boxplus \mu_2}(z))$ , then the relation satisfied by the  $R$ -transforms can be re-written as

$$(1) \quad \omega_{a_1}(z) + \omega_{a_2}(z) - z = \frac{1}{G_{\mu_1 \boxplus \mu_2}(z)} = \frac{1}{G_{\mu_1}(\omega_{a_1}(z))} = \frac{1}{G_{\mu_2}(\omega_{a_2}(z))}.$$

It has been shown that the functions  $\omega_{a_j}$ ,  $j = 1, 2$ , called the *subordination functions*, extend analytically as self-maps of the complex upper half-plane  $\mathbb{C}^+$  and Equation (1) holds for all  $z \in \mathbb{C}^+$  (see [17, 9, 4]). Of course, a similar relation holds for  $R_{\nu_j}(w)$ ,  $G_{\nu_j}(w)$  and  $\omega_{b_j}(w)$ ,  $j = 1, 2$ .

For the measure  $\eta_j$  (which is the distribution of the pair  $(a_j, b_j)$  with respect to  $\varphi$ ), Voiculescu introduces in [23, Theorem 2.1] the function

$$(2) \quad R_{(a_j, b_j)}(z, w) = R_{\eta_j}(z, w) = 1 + zR_{\mu_j}(z) + wR_{\nu_j}(w) - \frac{zw}{G_{\eta_j}(K_{\mu_j}(z), K_{\nu_j}(w))},$$

for  $z, w$  in a small enough bi-disk centred at zero (also see [14, Section 7.2]). Observe first that this function is indeed well-defined, including at zero, since

$$\lim_{w \rightarrow 0} \lim_{z \rightarrow 0} \frac{G_{\eta_j}(K_{\mu_j}(z), K_{\nu_j}(w))}{zw} = \lim_{w \rightarrow 0} \frac{G_{\nu_j}(K_{\nu_j}(w))}{w} = 1.$$

The limits can clearly be permuted. In particular,  $R_{\eta_j}(0, 0) = 0$ . Theorem 2.1 combined with Section 1.2 from [23] provide the following:

$$(3) \quad R_{\eta_1}(z, w) + R_{\eta_2}(z, w) = R_{\eta_1 \boxplus \eta_2}(z, w), \quad |z| + |w| \text{ sufficiently small.}$$

Given the linearizing property of the one-variable  $R$ -transform, this is equivalent to

$$\frac{zw}{G_{\eta_1}(K_{\mu_1}(z), K_{\nu_1}(w))} + \frac{zw}{G_{\eta_2}(K_{\mu_2}(z), K_{\nu_2}(w))} - 1 = \frac{zw}{G_{\eta_1 \boxplus \eta_2}(K_{\mu_1 \boxplus \mu_2}(z), K_{\nu_1 \boxplus \nu_2}(w))}.$$

This relation and Equation (1) allow us to write a formula for  $G_{\eta_1 \boxplus \eta_2}$  defined on all of  $(\mathbb{C} \setminus \sigma(a_1 + a_2)) \times (\mathbb{C} \setminus \sigma(b_1 + b_2))$  involving the subordination functions of free additive convolution. If we divide by  $zw$  and replace in the above  $z$  by  $G_{\mu_1 \boxplus \mu_2}(z)$  and  $w$  by  $G_{\nu_1 \boxplus \nu_2}(w)$ , then

$$(4) \quad \frac{1}{G_{\eta_1}(\omega_{a_1}(z), \omega_{b_1}(w))} + \frac{1}{G_{\eta_2}(\omega_{a_2}(z), \omega_{b_2}(w))} = \frac{1}{G_{\mu_1 \boxplus \mu_2}(z)G_{\nu_1 \boxplus \nu_2}(w)} + \frac{1}{G_{\eta_1 \boxplus \eta_2}(z, w)},$$

for any  $z \in \mathbb{C} \setminus \sigma(a_1 + a_2)$ ,  $w \in \mathbb{C} \setminus \sigma(b_1 + b_2)$ , as an equality of meromorphic functions (see [23, Remark 2.3] - in fact, [23, Lemma 2.6] states, if carefully read, precisely the relation above).

Most importantly for us, Voiculescu extended in [19] the analytic subordination results from above to selfadjoint random variables which are *free with amalgamation* over some subalgebra. We outline his result below.

Let  $(M, E, B)$  be an operator-valued  $W^*$ -noncommutative probability space, that is,  $B \subseteq M$  is a unital inclusion of (unital)  $W^*$ -algebras, and  $E: M \rightarrow B$  is a unit-preserving conditional expectation. Let  $X_1 = X_1^*$ ,  $X_2 = X_2^* \in M$  be free

over  $B$  with respect to  $E$ . Then for all  $n \in \mathbb{N}$ , there exists an analytic function  $\omega$  on  $M_n(B)$  such that

$$(E \otimes \text{Id}_{M_n(B)}) [(v - (X_1 + X_2) \otimes I_n)^{-1}] = (E \otimes \text{Id}_{M_n(B)}) [(\omega(v) - X_1 \otimes I_n)^{-1}],$$

for all  $v \in M_n(B)$  with strictly positive imaginary part or of inverse of sufficiently small norm. The function  $\omega$  increases the imaginary part of  $v$  if  $\Im v > 0$ .

The functions of the type  $(E \otimes \text{Id}_{M_n(B)}) [(v - X \otimes I_n)^{-1}]$  are natural extensions of the classical Cauchy transforms and share many of the properties of their classical, complex-valued, counterparts. Voiculescu showed in [18] that they allow the definition of  $B$ -valued  $R$ -transforms via the exact same procedure as for the complex-valued  $R$ -transforms, and that these  $R$ -transforms satisfy  $R_{X_1+X_2}(v) = R_{X_1}(v) + R_{X_2}(v)$  on a small neighbourhood of zero in  $B$ . In [21], he extended this relation to the amplification of the  $R$ -transform to  $M_n(B)$ ,  $n \in \mathbb{N}$ . An argument similar to the one used to prove (1) shows that the  $B$ -valued subordination functions satisfy precisely the same equation (1), but with variables  $v \in M_n(B)$ ,  $\Im v > 0$  instead of variables  $z \in \mathbb{C}^+$  (see [6] for details).

An operator-valued version of the analytic transforms of bi-freeness has been elaborated by one of us in [15]. We present a version of Equation (4) for operator-valued transforms. We consider a  $C^*$ - $B$ - $B$ -noncommutative probability space; the case when  $B$  is finite dimensional is of a special interest to us (see [15, Definitions 2.5 and 5.1] for details). An important difference from the case of  $\mathbb{C}$ -valued analytic transforms comes from the fact that a noncommutative algebra may receive a natural “opposite” structure. Thus, if analytic transforms of left random variables in a  $B$ - $B$ -noncommutative probability space coincide with Voiculescu’s analytic transforms introduced above, analytic transforms of the right random variables, while defined the same way (and thus having the same analytic properties), are viewed as being defined on (open subsets of)  $B^{\text{op}}$ , the algebra defined by the multiplication  $a \cdot_{\text{op}} a' = a'a$ . Thus, we are forced to add an  $\ell$  or an  $r$  to each analytic transform defined on  $B$ . In addition, the  $B$ - $B$ -valued equivalent of  $G_\eta(z, w)$  (or, more precisely, of  $G_\eta(z^{-1}, w^{-1})/zw$ ), becomes a function of three variables  $M_{(X,Y)}(b, c, d)$ , linear in  $c \in B$ , and for which  $b \in B$  is a “left” indeterminate and  $d \in B^{\text{op}}$  is a “right” indeterminate (whether  $c$  is viewed as a left or right indeterminate is irrelevant). More specifically, for a bi-random variable  $(X, Y)$  we define the partial moment generating function

$$M_{(X,Y)}(b, c, d) := \sum_{n,m \geq 0} E((L_b X)^n (R_d Y)^m R_c), \quad b, c, d \in B, \|b\|, \|d\| \text{ small.}$$

The moment generating functions of the left and right variables are  $M_X^\ell(b) = \sum_{n \geq 0} E((L_b X)^n)$  and  $M_X^r(d) = \sum_{n \geq 0} E((R_d X)^n)$ , respectively. For the purposes of this paper, the reader is invited to see  $L_b$  and  $R_d$  just as special ways of viewing the scalar algebras  $B$  and  $B^{\text{op}}$  embedded in the noncommutative probability space  $M$ . Thus, Voiculescu’s subordination relations from above are re-written in terms of the two moment generating functions as

$$G_{X_1+X_2}(b^{-1}) = M_{X_1+X_2}^\ell(b)b = M_{X_1}^\ell(\omega(b^{-1})^{-1})\omega(b^{-1})^{-1},$$

$$G_{Y_1+Y_2}(d^{-1}) = dM_{Y_1+Y_2}^r(d) = \omega(d^{-1})^{-1}M_{Y_1}^r(\omega(d^{-1})^{-1}),$$

respectively (the convention from [15] is slightly different from ours: the  $G(b)$  from [15] is  $G(b^{-1})$  here).

For our purposes, we prefer to view  $M_{(X,Y)}$  rather as an analytic function from  $B \times B$  with values in  $\mathcal{L}(B)$ , the space of continuous linear operators from  $B$  to itself. Viewed as such, we have  $M_{(X,Y)}(0, c, 0) = c$ , i.e.  $M_{(X,Y)}(0, \cdot, 0) = \text{Id}_B$ . Since the correspondence  $B \times B \ni (b, d) \mapsto M_{(X,Y)}(b, \cdot, d) \in \mathcal{L}(B)$  is analytic, we conclude that on a small enough norm-neighbourhood of  $(0, 0)$ , the element  $M_{(X,Y)}(b, \cdot, d) \in \mathcal{L}(B)$  is invertible as a linear map from  $B$  to  $B$ . We define  $\Psi_{(X,Y)}(b, \cdot, d) = b^{-1} M_{(X,Y)}(b, \cdot, d)^{(-1)} d^{-1} \in \mathcal{L}(B)$ , where  $M_{(X,Y)}(b, \cdot, d)^{(-1)}$  is the inverse of  $M_{(X,Y)}(b, \cdot, d)$  in  $\mathcal{L}(B)$ .

With these notations, the partial  $R$ -transform defined in [15, Section 5] is the analytic map  $B \times B \ni (b, d) \mapsto R_{(X,Y)}(b, \cdot, d) \in \mathcal{L}(B)$  uniquely determined on a neighbourhood of  $(0, 0)$  by the initial condition  $R_{(X,Y)}(0, \cdot, 0) = 0$  (the zero element in  $\mathcal{L}(B)$ ), and the functional equation

$$R_{(X,Y)}(M_X^\ell(b)b, c, dM_Y^r(d)) = M_X^\ell(b)c + cM_Y^r(d) - M_X^\ell(b)b\Psi_{(X,Y)}(b, c, d)dM_Y^r(d) - c.$$

The partial  $R$ -transform satisfies

$$R_{(X_1+X_2, Y_1+Y_2)}(b, c, d) = R_{(X_1, Y_1)}(b, c, d) + R_{(X_2, Y_2)}(b, c, d).$$

As before, we replace  $b$  by  $bM_{X_1+X_2}^\ell(b)$  and  $d$  by  $M_{Y_1+Y_2}^r(d)d$  in the  $R$ -transform relation to obtain

$$\begin{aligned} & M_{X_1+X_2}^\ell(b)c + cM_{Y_1+Y_2}^r(d) - M_{X_1+X_2}^\ell(b)b\Psi_{(X_1+X_2, Y_1+Y_2)}(b, c, d)dM_{Y_1+Y_2}^r(d) - c \\ &= R_{(X_1+X_2, Y_1+Y_2)}(M_{X_1+X_2}^\ell(b)b, c, dM_{Y_1+Y_2}^r(d)) \\ &= R_{(X_1, Y_1)}(M_{X_1+X_2}^\ell(b)b, c, dM_{Y_1+Y_2}^r(d)) + R_{(X_2, Y_2)}(M_{X_1+X_2}^\ell(b)b, c, dM_{Y_1+Y_2}^r(d)). \end{aligned}$$

The subordination relation provides

$$\begin{aligned} & R_{(X_j, Y_j)}(M_{X_1+X_2}^\ell(b)b, c, dM_{Y_1+Y_2}^r(d)) \\ &= R_{(X_j, Y_j)}(M_{X_j}^\ell(\omega_{X_j}(b^{-1})^{-1})\omega_{X_j}(b^{-1})^{-1}, c, \omega_{Y_j}(d^{-1})^{-1}M_{Y_j}^r(\omega_{Y_j}(d^{-1})^{-1})) \\ &= M_{X_j}^\ell(\omega_{X_j}(b^{-1})^{-1})c + cM_{Y_j}^r(\omega_{Y_j}(d^{-1})^{-1}) - c \\ &\quad - M_{X_j}^\ell(\omega_{X_j}(b^{-1})^{-1})\omega_{X_j}(b^{-1})^{-1} \\ &\quad \times \Psi_{(X_j, Y_j)}(\omega_{X_j}(b^{-1})^{-1}, c, \omega_{Y_j}(d^{-1})^{-1})\omega_{Y_j}(d^{-1})^{-1}M_{Y_j}^r(\omega_{Y_j}(d^{-1})^{-1}). \end{aligned}$$

This allows the following analogue of relation (4):

$$\begin{aligned} & M_{X_1+X_2}^\ell(b)c + cM_{Y_1+Y_2}^r(d) - M_{X_1+X_2}^\ell(b)b\Psi_{(X_1+X_2, Y_1+Y_2)}(b, c, d)dM_{Y_1+Y_2}^r(d) - c \\ &= M_{X_1}^\ell(\omega_{X_1}(b^{-1})^{-1})c + cM_{Y_1}^r(\omega_{Y_1}(d^{-1})^{-1}) - c \\ &\quad - M_{X_1}^\ell(\omega_{X_1}(b^{-1})^{-1})\omega_{X_1}(b^{-1})^{-1} \\ &\quad \times \Psi_{(X_1, Y_1)}(\omega_{X_1}(b^{-1})^{-1}, c, \omega_{Y_1}(d^{-1})^{-1})\omega_{Y_1}(d^{-1})^{-1}M_{Y_1}^r(\omega_{Y_1}(d^{-1})^{-1}) \\ &\quad + M_{X_2}^\ell(\omega_{X_2}(b^{-1})^{-1})c + cM_{Y_2}^r(\omega_{Y_2}(d^{-1})^{-1}) - c \\ &\quad - M_{X_2}^\ell(\omega_{X_2}(b^{-1})^{-1})\omega_{X_2}(b^{-1})^{-1} \\ (5) \quad & \times \Psi_{(X_2, Y_2)}(\omega_{X_2}(b^{-1})^{-1}, c, \omega_{Y_2}(d^{-1})^{-1})\omega_{Y_2}(d^{-1})^{-1}M_{Y_2}^r(\omega_{Y_2}(d^{-1})^{-1}). \end{aligned}$$

Furthermore, recalling that  $G_X(b) = M_X^\ell(b^{-1})b^{-1}$ , the  $B$ -valued version of (1) written under the form

$$G_{X_1+X_2}(b^{-1})b^{-1} = G_{X_1}(\omega_{X_1}(b^{-1}))\omega_{X_1}(b^{-1}) + G_{X_2}(\omega_{X_2}(b^{-1}))\omega_{X_2}(b^{-1}) - 1$$

allows us to simplify the above to

$$\begin{aligned}
& M_{X_1+X_2}^\ell(b) \Psi_{(X_1+X_2, Y_1+Y_2)}(b, c, d) d M_{Y_1+Y_2}^r(d) + c \\
&= M_{X_1}^\ell(\omega_{X_1}(b^{-1})^{-1}) \omega_{X_1}(b^{-1})^{-1} \\
&\quad \times \Psi_{(X_1, Y_1)}(\omega_{X_1}(b^{-1})^{-1}, c, \omega_{Y_1}(d^{-1})^{-1}) \omega_{Y_1}(d^{-1})^{-1} M_{Y_1}^r(\omega_{Y_1}(d^{-1})^{-1}) \\
&\quad + M_{X_2}^\ell(\omega_{X_2}(b^{-1})^{-1}) \omega_{X_2}(b^{-1})^{-1} \\
(6) \quad &\quad \times \Psi_{(X_2, Y_2)}(\omega_{X_2}(b^{-1})^{-1}, c, \omega_{Y_2}(d^{-1})^{-1}) \omega_{Y_2}(d^{-1})^{-1} M_{Y_2}^r(\omega_{Y_2}(d^{-1})^{-1}).
\end{aligned}$$

Since we usually prefer to deal with  $G$  rather than  $M$ , we perform in the above the changes of variable  $b \mapsto b^{-1}$  and  $d \mapsto d^{-1}$  in order to obtain

$$\begin{aligned}
& G_{X_1+X_2}(b) \Psi_{(X_1+X_2, Y_1+Y_2)}(b^{-1}, c, d^{-1}) G_{Y_1+Y_2}(d) + c \\
&= G_{X_1}(\omega_{X_1}(b)) \Psi_{(X_1, Y_1)}(\omega_{X_1}(b)^{-1}, c, \omega_{Y_1}(d)^{-1}) G_{Y_1}(\omega_{Y_1}(d)) \\
(7) \quad &\quad + G_{X_2}(\omega_{X_2}(b)) \Psi_{(X_2, Y_2)}(\omega_{X_2}(b)^{-1}, c, \omega_{Y_2}(d)^{-1}) G_{Y_2}(\omega_{Y_2}(d)),
\end{aligned}$$

a precise analogue of equation (4) (the parallel is made more obvious by the fact that  $\Psi(b^{-1}, \cdot, d^{-1}) = G(b, \cdot, d)^{(-1)}$  if  $G(b^{-1}, b^{-1}cd^{-1}, d^{-1}) := M(b, c, d)$ ). This relation holds for all  $b, d$  for which the functions  $\Psi$  are defined, hence at least for  $b, d$  satisfying the requirement that  $\|b^{-1}\|$  and  $\|d^{-1}\|$  are sufficiently small. This is an open set, so that the above relation does determine  $\Psi_{(X_1+X_2, Y_1+Y_2)}$  uniquely from the knowledge of  $\Psi_{(X_j, Y_j)}$ ,  $j = 1, 2$ , and of the free additive convolution of operator-valued distributions. We record next a slightly different version of (7), which resembles the scalar reduced partial  $R$ -transform. Its validity follows trivially from (7).

$$\begin{aligned}
& \Psi_{(X_1+X_2, Y_1+Y_2)}(K_{X_1+X_2}(b)^{-1}, c, K_{Y_1+Y_2}(d)^{-1}) - b^{-1}cd^{-1} = \\
& \Psi_{(X_1, Y_1)}(K_{X_1}(b)^{-1}, c, K_{Y_1}(d)^{-1}) - b^{-1}cd^{-1} \\
(8) \quad & \quad + \Psi_{(X_2, Y_2)}(K_{X_2}(b)^{-1}, c, K_{Y_2}(d)^{-1}) - b^{-1}cd^{-1},
\end{aligned}$$

The transforms  $G, M, \Psi$  and  $R$  do not fully characterize the joint distribution of  $(X, Y)$ , but only the “band moments.” However, this will suffice for our main purpose of studying general distributions of bi-partite bi-free random variables. In this context, our interest is mainly in bi-freeness with amalgamation over finite dimensional algebras. The construction providing an  $M_n(\mathbb{C})$ - $M_n(\mathbb{C})$ -noncommutative probability space from a classical noncommutative probability space  $(\mathcal{A}, \varphi)$  is the following (see [14, Section 6] or [15, Section 4]). First, one defines the left and right actions

$$L_b(T) = \left[ \sum_{k=1}^n b_{ik} T_{kj} \right]_{i,j=1}^n, \quad R_d(T) = \left[ \sum_{k=1}^n d_{kj} T_{ik} \right]_{i,j=1}^n,$$

for all  $b, d \in M_n(\mathbb{C})$ ,  $T \in \mathcal{L}(M_n(\mathcal{A}))$ . Both  $L_b$  and  $R_d$  are indeed bounded linear maps on  $M_n(\mathcal{A})$ . The correspondences  $b \mapsto L_b$  and  $d \mapsto R_d$  are algebra homomorphisms from  $M_n(\mathbb{C})$  and  $M_n(\mathbb{C})^{\text{op}}$ , respectively, into  $\mathcal{L}(M_n(\mathcal{A}))$ . We define the right algebra  $\mathcal{L}(M_n(\mathcal{A}))_r$  as the set

$$\{Z \in \mathcal{L}(M_n(\mathcal{A})) : ZL_b = L_bZ \text{ for all } b \in M_n(\mathbb{C})\},$$

and  $\mathcal{L}(M_n(\mathcal{A}))_\ell$  the same way, but with  $L$  replaced by  $R$ . One embeds  $M_n(\mathcal{A})$  in  $\mathcal{L}(M_n(\mathcal{A}))_r$  via

$$Z \mapsto R(Z), \quad R(Z)(T) = \left[ \sum_{k=1}^n Z_{kj} T_{ik} \right]_{i,j=1}^n \quad \text{for all } T \in M_n(\mathcal{A}),$$

and in  $\mathcal{L}(M_n(\mathcal{A}))_\ell$  via

$$Z \mapsto L(Z), \quad L(Z)(T) = \left[ \sum_{k=1}^n Z_{ik} T_{kj} \right]_{i,j=1}^n \quad \text{for all } T \in M_n(\mathcal{A}).$$

The map  $Z \mapsto L(Z)$  is an injective algebra  $*$ -homomorphism from  $M_n(\mathcal{A})$  into  $\mathcal{L}(M_n(\mathcal{A}))$ , and  $Z \mapsto R(Z)$  from  $M_n(\mathcal{A}^{\text{op}})^{\text{op}}$  into  $\mathcal{L}(M_n(\mathcal{A}))$ . The conditional expectation  $E_n: \mathcal{L}(M_n(\mathcal{A})) \rightarrow M_n(\mathbb{C})$  defined by  $E_n[W] = [\varphi(W(I_n)_{ij})]_{i,j=1}^n$  satisfies  $E_n[L(Z)] = [\varphi(Z_{ij})]_{i,j=1}^n$  and  $E_n[R(Z)] = [\varphi(Z_{ij})]_{i,j=1}^n$ . That is, the distribution of  $Z \in M_n(\mathcal{A})$  with respect to  $\varphi \otimes \text{Id}_{M_n(\mathbb{C})}$  is the same as the distribution of  $L(Z)$  (resp.  $R(Z)$ ) with respect to  $E_n$ . We note that  $R_d R(Y) = R(Yd)$ ,  $L_b L(X) = L(bX)$ , and  $R_{I_n} = L_{I_n} = \text{Id}_{M_n(\mathcal{A})}$ . Then, the map  $M_{(X,Y)}$  defined above is written as

$$\begin{aligned} M_{(X,Y)}(b, c, d) &= E_n [(1 - L_b L(X))^{-1} (1 - R_d R(Y))^{-1} R_c] \\ &= (\varphi \otimes \text{Id}_{M_n(\mathbb{C})}) [(1 - L(bX))^{-1} (1 - R(Yd))^{-1} R(c)(I_n)] \\ &= (\varphi \otimes \text{Id}_{M_n(\mathbb{C})}) [L((1 - bX)^{-1}) R((1 - Yd)^{-1}) R(c)(I_n)] \\ &= (\varphi \otimes \text{Id}_{M_n(\mathbb{C})}) [L((1 - bX)^{-1}) R(c(1 - Yd)^{-1})(I_n)] \\ &= (\varphi \otimes \text{Id}_{M_n(\mathbb{C})}) [L((1 - bX)^{-1}) (c(1 - Yd)^{-1})] \\ &= (\varphi \otimes \text{Id}_{M_n(\mathbb{C})}) [(1 - bX)^{-1} c(1 - Yd)^{-1}]. \end{aligned}$$

Thus, vitally for us, the band-moment generating function for a pair of faces  $(L(X), R(Y))$  coincides with the band moment generating function of  $(X, Y)$ . This allows us to extend the map  $M_{(X,Y)}(b, \cdot, d)$ , as its scalar-valued analogue, to  $\{b \in M_n(\mathbb{C}): \pm \Im b > 0\} \times \{d \in M_n(\mathbb{C})^{\text{op}}: \pm \Im d > 0\}$ .

To conclude this section, we state a particular case of [14, Theorem 6.3.1] (alternatively see [15, Theorem 4.1]):

**Lemma 2.1.** *Let  $(\mathcal{A}, \varphi)$  be a noncommutative probability space and  $n \in \mathbb{N}$ . Assume that  $(a_1, b_1)$  and  $(a_2, b_2)$  are bi-free with respect to  $\varphi$ . Then  $(L(X_1), R(Y_1))$  and  $(L(X_2), R(Y_2))$  are bi-free with amalgamation over  $M_n(\mathbb{C})$  whenever  $X_j \in M_n(\mathbb{C}[a_j])$ ,  $Y_j \in M_n(\mathbb{C}[b_j])$ ,  $j = 1, 2$ .*

### 3. BI-FREE ANALYTIC SUBORDINATION

In this section we establish our main subordination result for scalar-valued bi-free random variables. Fix a  $*$ -probability space  $(\mathcal{A}, \varphi)$  and two pairs  $(a_1, b_1), (a_2, b_2) \in \mathcal{A}^2$  that are bi-free with respect to  $\varphi$  and bi-partite. Thus,  $a_j$  (respectively  $b_j$ ) is the left (respectively right) variable in the pair  $(a_j, b_j)$ , and  $a_j b_j = b_j a_j$ ,  $j = 1, 2$ . (Many of the computations below are valid under weaker hypotheses. In many circumstances, our computations with analytic transforms also hold for operator-valued bi-free pairs of random variables. We indicate below those cases in which this extension is valid.)

Define  $X_j \in M_2(\mathcal{A})$  by  $X_j = \begin{bmatrix} a_j & 0 \\ 0 & b_j \end{bmatrix}$ ,  $j = 1, 2$ . In the context we will consider below, it is important that the left face is in the upper left, and the right face in the lower right, corner. We define the conditional expectation  $M_2(\varphi) = \varphi \otimes \text{Id}_{M_2(\mathbb{C})}$  from  $M_2(\mathcal{A})$  onto  $M_2(\mathbb{C})$ . With respect to this expectation, we define  $G_{X_j}(v) = E[(v - X_j)^{-1}]$  for  $v \in M_2(\mathbb{C})$  such that  $\Im v > 0$  or  $\|v^{-1}\| < \|X_j\|^{-1}$ . As usual, the  $R$ -transform is defined via the functional equation  $G_{X_j}(v^{-1} + R_{X_j}(v)) = v$ . As before, denote  $K_{X_j}(v) = v^{-1} + R_{X_j}(v)$ . We restrict all of these functions to upper triangular matrices in  $M_2(\mathbb{C})$ . Direct computation using the Schur complement yields

$$\begin{aligned} G_{X_j} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) &= M_2(\varphi) \begin{bmatrix} (z - a_j)^{-1} & -(z - a_j)^{-1} \zeta (w - b_j)^{-1} \\ 0 & (w - b_j)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \varphi((z - a_j)^{-1}) & -\varphi((z - a_j)^{-1} \zeta (w - b_j)^{-1}) \\ 0 & \varphi((w - b_j)^{-1}) \end{bmatrix} \\ &= \begin{bmatrix} G_{\mu_j}(z) & -\zeta G_{\eta_j}(z, w) \\ 0 & G_{\nu_j}(w) \end{bmatrix}. \end{aligned}$$

(First two equalities hold for operator-valued random variables  $(a_j, b_j)$ , with the map  $\varphi((z - a_j)^{-1} \zeta (w - b_j)^{-1})$  replaced by  $\zeta \mapsto M_{(a_j, b_j)}(z^{-1}, z^{-1} \zeta w^{-1}, w^{-1}) = G_{(a_j, b_j)}(z, \zeta, w)$  - see Section 2.) The compositional inverse of an analytic map that maps upper triangular matrices onto upper triangular matrices preserves itself upper triangular matrices. That is,  $K_{X_j} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) = \begin{bmatrix} K_{a_j}(z) & -h(z, \zeta, w) \\ 0 & K_{b_j}(w) \end{bmatrix} = \begin{bmatrix} K_{\mu_j}(z) & -h(z, \zeta, w) \\ 0 & K_{\nu_j}(w) \end{bmatrix}$  for some function  $h(z, \zeta, w)$ . Then, by the above formula,

$$\begin{aligned} \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} &= G_{X_j} \left( \begin{bmatrix} K_{a_j}(z) & -h(z, \zeta, w) \\ 0 & K_{b_j}(w) \end{bmatrix} \right) \\ &= \begin{bmatrix} G_{a_j}(K_{a_j}(z)) & \varphi((K_{a_j}(z) - a_j)^{-1} h(z, \zeta, w) (K_{b_j}(w) - b_j)^{-1}) \\ 0 & G_{b_j}(K_{b_j}(w)) \end{bmatrix} \\ &= \begin{bmatrix} G_{a_j}(K_{a_j}(z)) & G_{(a_j, b_j)}(K_{a_j}(z), h(z, \zeta, w), K_{b_j}(w)) \\ 0 & G_{b_j}(K_{b_j}(w)) \end{bmatrix} \\ &= \begin{bmatrix} G_{\mu_j}(K_{\mu_j}(z)) & h(z, \zeta, w) G_{\eta_j}(K_{\mu_j}(z), K_{\nu_j}(w)) \\ 0 & G_{\nu_j}(K_{\nu_j}(w)) \end{bmatrix}. \end{aligned}$$

(Again, the first three equalities hold for operator-valued variables.) The (1, 2) entry shows that  $h(z, \zeta, w)$  is linear in  $\zeta$ , and has as inverse the linear map  $\xi \mapsto G_{(a_j, b_j)}(K_{a_j}(z), \xi, K_{b_j}(w))$ . Thus,

$$\begin{aligned} h(z, \cdot, w) &= G_{(a_j, b_j)}(K_{a_j}(z), \cdot, K_{b_j}(w))^{\langle -1 \rangle} \\ &= K_{\mu_j}(z) M_{(a_j, b_j)}(K_{\mu_j}(z)^{-1}, \cdot, K_{\nu_j}(w)^{-1})^{\langle -1 \rangle} K_{\nu_j}(w) \\ &= \Psi_{(a_j, b_j)}(K_{\mu_j}(z)^{-1}, \cdot, K_{\nu_j}(w)^{-1}), \end{aligned}$$

and hence

$$h(z, \zeta, w) = \Psi_{(a_j, b_j)}(K_{\mu_j}(z)^{-1}, \zeta, K_{\nu_j}(w)^{-1}) = \frac{\zeta}{G_{\eta_j}(K_{\mu_j}(z), K_{\nu_j}(w))}.$$



(The last quantity above only makes sense for scalar-valued variables.) In particular,

$$\begin{aligned}
 R_{X_j} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) &= \begin{bmatrix} K_{\mu_j}(z) - z^{-1} & -h(z, \zeta, w) + z^{-1}\zeta w^{-1} \\ 0 & K_{\nu_j}(w) - w^{-1} \end{bmatrix} \\
 &= \begin{bmatrix} R_{\mu_j}(z) & z^{-1}\zeta w^{-1} - \Psi_{(a_j, b_j)}(K_{\mu_j}(z)^{-1}, \zeta, K_{\nu_j}(w)^{-1}) \\ 0 & R_{\nu_j}(w) \end{bmatrix} \\
 (9) \quad &= \begin{bmatrix} R_{\mu_j}(z) & \zeta \left( \frac{1}{zw} - \frac{1}{G_{\eta_j}(K_{\mu_j}(z), K_{\nu_j}(w))} \right) \\ 0 & R_{\nu_j}(w) \end{bmatrix}
 \end{aligned}$$

(with the first two equalities making sense for operator-valued variables). Observe that in all the computations this far we have not used the fact that  $a_j b_j = b_j a_j$ . If we agree to consider only the band moments of type  $\varphi(LR)$  ( $L$  and  $R$  being monomials in the left, respectively right, variable), all the above computations remain valid, with the possible difference that  $\varphi((z - a_j)^{-1}(w - b_j)^{-1})$  might not be the Cauchy transform of a probability measure  $\eta_j$  on  $\mathbb{R}^2$ . We record our conclusion in the following

**Lemma 3.1.** *Let  $(\mathcal{A}, \varphi)$  be a  $*$ -probability space and  $(a, b) \in \mathcal{A}^2$  be a two-faced pair of noncommutative random variables. Define the  $M_2(\mathbb{C})$ -valued random variable  $X = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_2(\mathcal{A})$ . Then*

$$R_X \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) = \begin{bmatrix} R_a(z) & \frac{\zeta}{zw} (R_{(a,b)}(z, w) - zR_a(z) - wR_b(w)) \\ 0 & R_b(w) \end{bmatrix}.$$

*In particular, if  $(a_1, b_1), (a_2, b_2) \in \mathcal{A}^2$  are bi-free with respect to  $\varphi$ , then, with the notations from the beginning of this section,*

$$(10) \quad R_{X_1 + X_2} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) = R_{X_1} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) + R_{X_2} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right)$$

*for all  $z, w \in \mathbb{C}$  of sufficiently small absolute value and all  $\zeta \in \mathbb{C}$ .*

(Lemma 3.1 holds for operator-valued random variables in a  $B$ - $B$ -valued  $*$ -noncommutative probability space  $M$ . More precisely, if  $(a, b) \in M^2$  is a two-faced pair of  $B$ -valued random variables, then, defining  $X = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in M_2(M)$ , we have

$$R_X \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) = \begin{bmatrix} R_a(z) & z^{-1}\zeta w^{-1} - \Psi_{(a,b)}(K_a(z)^{-1}, \zeta, K_b(w)^{-1}) \\ 0 & R_b(w) \end{bmatrix},$$

where  $\Psi_{(a,b)}$  has been defined in Section 2, and Equation (10) also holds in this context.)

We would like to emphasize again that  $R_X$  above denotes the  $R$ -transform of the  $M_2(\mathbb{C})$  (or  $M_2(B)$ )-valued random variable  $X$ , as introduced in [18]. Thus, relation (10) implies that  $X_1$  and  $X_2$  “mimic” freeness in terms of the relations between their analytic transforms when restricted to upper triangular matrices. More specifically:

**Remark 3.2.** Let  $(\mathcal{A}, \varphi)$  be a  $W^*$ -noncommutative probability space. Assume that  $(a_1, b_1)$  and  $(a_2, b_2)$  are selfadjoint and bi-free with respect to  $\varphi$ . Define

$X_j = \begin{bmatrix} a_j & 0 \\ 0 & b_j \end{bmatrix}$ ,  $j = 1, 2$  to be two selfadjoint random variables in the operator-valued noncommutative probability space  $(M_2(\mathcal{A}), \varphi \otimes \text{Id}_{M_2(\mathbb{C})}, M_2(\mathbb{C}))$ . Consider two random variables  $Y_1, Y_2$  which are free with respect to  $M_2(\varphi) := \varphi \otimes \text{Id}_{M_2(\mathbb{C})}$ , and such that the  $*$ -distribution of  $X_j$  and  $Y_j$  with respect to  $M_2(\varphi)$  coincide for  $j = 1, 2$ . Then the restrictions of the Cauchy and  $R$ -transforms of  $X_1, X_2, X_1 + X_2$  and  $Y_1, Y_2, Y_1 + Y_2$ , respectively, to the upper triangular  $2 \times 2$  complex matrices with positive imaginary part and to the set of upper triangular  $2 \times 2$  complex matrices having inverse of small norm coincide. In particular, if  $\omega_{Y_j}$  is the  $M_2(\mathbb{C})$ -valued subordination function satisfying  $G_{Y_j} \circ \omega_{Y_j} = G_{Y_1+Y_2}$ , then

$$(G_{X_j \circ \omega_{Y_j}}) \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right)^{-1} = G_{X_1+X_2} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right)^{-1} = (\omega_{Y_1} + \omega_{Y_2}) \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) - \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix}$$

for all  $z, w \in \mathbb{C}^+$ ,  $\zeta \in \mathbb{C}$ ,  $j = 1, 2$ . (All relations above hold as well for operator-valued variables.)

In light of Lemma 3.1 and Equation (1), the proof of the above remark is obvious as soon as one accounts for the fact that the  $M_2(\mathbb{C})$ -valued distribution of the variable  $X$  determines the  $M_2(\mathbb{C})$ -valued Cauchy transform of  $X$ . This remark allows us to recover Equation (4). Indeed, as it follows from [6, Theorem 2.7] that the functions  $\omega_{Y_j}$  map upper triangular matrices to upper triangular matrices, we have that  $\omega_{Y_j} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) = \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix}$ . Since  $X_j$  and  $Y_j$  have the same distribution and  $X_j$  is diagonal and selfadjoint, so must be  $Y_j$ , and its diagonal entries must have the same (joint) distribution as  $(a_j, b_j)$ . Thus, we obtain that

$$\begin{aligned} G_{Y_1+Y_2} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) &= G_{X_j} \left( \begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix} \right) \\ &= \begin{bmatrix} \varphi((f_1 - a_j)^{-1}) & -\varphi((f_1 - a_j)^{-1} f_2 (f_3 - b_j)^{-1}) \\ 0 & \varphi((f_3 - b_j)^{-1}) \end{bmatrix}, \end{aligned}$$

which guarantees that  $f_1 = \omega_{a_j}(z)$ ,  $f_3 = \omega_{b_j}(w)$ . Using again the previous remark, we obtain

$$\begin{aligned} f_2 &= \Psi_{(a_j, b_j)}(\omega_{a_1}(z)^{-1}, \varphi((z - a_1 - a_2)^{-1} \zeta (w - b_1 - b_2)^{-1}), \omega_{b_1}(w)^{-1}) \\ &= \frac{\zeta \varphi((z - a_1 - a_2)^{-1} (w - b_1 - b_2)^{-1})}{\varphi((\omega_{a_j}(z) - a_j)^{-1} (\omega_{b_j}(z) - b_j)^{-1})}. \end{aligned}$$

Again, the first equality is true for operator-valued random variables. Replacing this in the upper triangular matrix-valued analogue (provided above) of (1) provides Equation (4). (Observe that, while in order to obtain a relation between Cauchy transforms of measures in the plane, we need to assume that  $a_j$  and  $b_j$  commute, the formal calculations above hold even in the absence of this hypothesis.)

**Example 3.3.** Relation (10) does not extend to arbitrary  $2 \times 2$  matrices  $\begin{bmatrix} z & \zeta \\ \zeta' & w \end{bmatrix}$ . In other words,  $X_1$  and  $X_2$  are not necessarily free with amalgamation over  $M_2(\mathbb{C})$ , as it can be seen by computing some moments. Consider

$$M_2(\varphi) \left( \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} \right) = \begin{bmatrix} 0 & \varphi(a_1 a_2 b_2 b_1) \\ 0 & 0 \end{bmatrix}.$$

If  $(a_1, b_1)$  and  $(a_2, b_2)$  are bi-free with respect to  $\varphi$ , then

$$(11) \quad \varphi(a_1 a_2 b_2 b_1) = \varphi(a_1 b_1) \varphi(a_2) \varphi(b_2) + \varphi(a_2 b_2) \varphi(a_1) \varphi(b_1) - \varphi(a_1) \varphi(a_2) \varphi(b_1) \varphi(b_2).$$

On the other hand, if  $\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}$  and  $\begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$  were free with amalgamation over  $M_2(\mathbb{C})$ , then we would have

$$M_2(\varphi) \left( \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} \right)_{1,2} = \varphi(a_1 b_1) \varphi(a_2 b_2),$$

which is generally different from the right hand-side of (11).

The main result of this section follows easily from the above considerations, Remark 3.2 and Equation (7).

**Proposition 3.4.** *Let  $(\mathcal{A}, \varphi, B)$  be a  $B$ - $B^*$ -probability space, for a  $C^*$ -algebra  $B$ . Assume that  $(a_1, b_1)$  and  $(a_2, b_2)$  are selfadjoint random variables in  $\mathcal{A}$  which are bi-free over  $B$  with respect to  $\varphi$ . Denote  $X_j = \begin{bmatrix} a_j & 0 \\ 0 & b_j \end{bmatrix}$ ,  $j = 1, 2$ . Then*

$$\begin{aligned} M_2(\varphi) \left[ \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} - X_1 - X_2 \right)^{-1} \right] &= M_2(\varphi) \left[ \left( \begin{bmatrix} \omega_{a_1}(z) & \Pi_1(z, \zeta, w) \\ 0 & \omega_{b_1}(w) \end{bmatrix} - X_1 \right)^{-1} \right] \\ &= \begin{bmatrix} G_{a_1}(\omega_{a_1}(z)) & G_{(a_1, b_1)}(\omega_{a_1}(z), \Pi_1(z, \zeta, w), \omega_{b_1}(w)) \\ 0 & G_{b_1}(\omega_{b_1}(w)) \end{bmatrix}, \end{aligned}$$

where  $\Pi_1(z, \cdot, w)$  is the composition of the following linear maps depending analytically on  $z, w$ :

$$\begin{aligned} \Pi_1(z, \zeta, w) &= [G_{(a_1, b_1)}(\omega_{a_1}(z), \cdot, \omega_{b_1}(w)) + G_{(a_2, b_2)}(\omega_{a_2}(z), \cdot, \omega_{b_2}(w)) - \\ &\quad G_{(a_2, b_2)}(\omega_{a_2}(z), G_{a_1}(\omega_{a_1}(z))^{-1} G_{(a_1, b_1)}(\omega_{a_1}(z), \cdot, \omega_{b_1}(w)) G_{b_1}(\omega_{b_1}(w))^{-1}, \omega_{b_2}(w))]^{(-1)} \\ &\quad \circ G_{(a_2, b_2)}(\omega_{a_2}(z), \zeta, \omega_{b_2}(w)). \end{aligned}$$

If  $B = \mathbb{C}$ , then  $\varphi$  is a state and  $\Pi_1$  has the simpler formula

$$\Pi_1(z, \zeta, w) = \frac{\frac{\zeta}{G_{(a_1, b_1)}(\omega_{a_1}(z), \omega_{b_1}(w))}}{\frac{1}{G_{(a_1, b_1)}(\omega_{a_1}(z), \omega_{b_1}(w))} + \frac{1}{G_{(a_2, b_2)}(\omega_{a_2}(z), \omega_{b_2}(w))} - \frac{1}{G_{a_1}(\omega_{a_1}(z)) G_{b_1}(\omega_{b_1}(w))}}$$

for all  $z, w \in \mathbb{C}^+$ ,  $\zeta \in \mathbb{C}$ . Here  $\omega_{a_1}$  and  $\omega_{b_1}$  are the subordination functions introduced in Section 2, Equation (1), and  $M_2(\varphi)$  is the conditional expectation onto  $M_2(B)$  given by  $\varphi \otimes \text{tr}_2$ .

The following proposition establishes some analytic properties of the Cauchy transform of the bi-free additive convolution of two probability measures in  $\mathbb{R}^2$  that will help us prove a converse of the characterization of bi-free extreme values of Voiculescu.

**Proposition 3.5.** *With the notation from Proposition 3.4 and from Section 2, we have*

$$\Im G_{a_j}(z) \Im G_{b_j}(w) \geq \Im z \Im w |G_{(a_j, b_j)}(z, w)|^2,$$

and

$$\Im \omega_{a_j}(z) \Im \omega_{b_j}(w) |G_{\eta_j}(\omega_{a_j}(z), \omega_{b_j}(w))|^2 \geq \Im z \Im w |G_{\eta_1 \boxplus \eta_2}(z, w)|^2, \quad z, w \in \mathbb{C}^+.$$

*Proof.* The proof is based on a simple trick, which is a particular case of [3, Proposition 3.1]. Observe that  $\Im \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} > 0$  if and only if  $\Im z > 0, \Im w > 0$  and  $4\Im z \Im w > |\zeta|^2$ . As  $G_{X_j}$  maps elements from  $M_2(\mathbb{C})$  with positive imaginary part into elements from  $M_2(\mathbb{C})$  with negative imaginary part, it follows that  $4\Im G_{a_j}(z) \Im G_{b_j}(w) > |\zeta|^2 |G_{(a_j, b_j)}(z, w)|^2$  whenever  $\Im \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} > 0$ . Letting  $|\zeta|$  tend to  $2\sqrt{\Im z \Im w}$  from below yields

$$\Im G_{a_j}(z) \Im G_{b_j}(w) \geq \Im z \Im w |G_{(a_j, b_j)}(z, w)|^2, \quad z, w \in \mathbb{C}^+.$$

The second relation follows from the fact that  $\Im \omega_{X_j} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) > 0$  whenever  $\Im \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} > 0$ , and the fact that

$$\omega_{X_j} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) = \begin{pmatrix} \omega_{a_j}(z) & \Pi_j(z, \zeta, w) \\ 0 & \omega_{b_j}(w) \end{pmatrix}.$$

Indeed, it is known from [19] that the maps  $\omega$  introduced in (1) map the set of elements with positive imaginary part into itself. Proposition 3.4 and Remark 3.2 guarantee the validity of the above displayed relation, and, together with (4), provide the equality  $\Pi_j(z, \zeta, w) = \frac{\zeta G_{\eta_1 \boxplus \eta_2}(z, w)}{G_{\eta_j}(\omega_{a_j}(z), \omega_{b_j}(w))}$ . We obtain again that  $\Im z > 0, \Im w > 0$  and  $4\Im z \Im w > |\zeta|^2$  imply together that  $4\Im \omega_{a_j}(z) \Im \omega_{b_j}(w) > \left| \frac{\zeta G_{\eta_1 \boxplus \eta_2}(z, w)}{G_{\eta_j}(\omega_{a_j}(z), \omega_{b_j}(w))} \right|^2$ . Letting  $|\zeta|$  tend to  $2\sqrt{\Im z \Im w}$  from below allows us to conclude.  $\square$

The following corollary is the converse of [24, Theorem 2.1].

**Corollary 3.6.** *Assume that  $(a_1, b_1)$  and  $(a_2, b_2)$  are bi-free bi-partite selfadjoint random variables. We denote by  $\eta_j$  the distribution of  $(a_j, b_j)$  and by  $\mu_j$  and  $\nu_j$  its first and second marginal, respectively. Assume that there exists a point  $(\xi, \zeta) \in \mathbb{R}^2$  such that  $(\eta_1 \boxplus \eta_2)(\{(\xi, \zeta)\}) > 0$ . Then there exist  $(\xi_j, \zeta_j) \in \mathbb{R}^2, j = 1, 2$ , such that  $(\xi_1 + \xi_2, \zeta_1 + \zeta_2) = (\xi, \zeta)$  and  $\eta_j(\{(\xi_j, \zeta_j)\}) > 0$ . Moreover,*

$$1 + \frac{(\mu_1 \boxplus \mu_2)(\{\xi\})(\nu_1 \boxplus \nu_2)(\{\zeta\})}{(\eta_1 \boxplus \eta_2)(\{(\xi, \zeta)\})} = \frac{\mu_1(\{\xi_1\})\nu_1(\{\zeta_1\})}{\eta_1(\{(\xi_1, \zeta_1)\})} + \frac{\mu_2(\{\xi_2\})\nu_2(\{\zeta_2\})}{\eta_2(\{(\xi_2, \zeta_2)\})}.$$

*Proof.* Observe first that

$$(\mu_1 \boxplus \mu_2)(\{\xi\}) = \int_{\mathbb{R}} \mathbf{1}_{\{\xi\} \times \mathbb{R}}(s, t) d(\eta_1 \boxplus \eta_2)(s, t) \geq (\eta_1 \boxplus \eta_2)(\{(\xi, \zeta)\}),$$

with a similar result for  $\nu_1 \boxplus \nu_2$ . Thus, [7, Theorem 7.4] indicates that  $\mu_j$  and  $\nu_j$  all have atoms. More precise, there are  $\xi_j, \zeta_j \in \mathbb{R}, j = 1, 2$ , such that

- (1)  $1 < \mu_1(\{\xi_1\}) + \mu_2(\{\xi_2\}) = (\mu_1 \boxplus \mu_2)(\{\xi\}) + 1$ ;
- (2)  $1 < \nu_1(\{\zeta_1\}) + \nu_2(\{\zeta_2\}) = (\nu_1 \boxplus \nu_2)(\{\zeta\}) + 1$ ;
- (3)  $\xi_1 + \xi_2 = \xi$ ;
- (4)  $\zeta_1 + \zeta_2 = \zeta$ .

According to the same article [7], from this it follows that  $\omega_{a_j}(iy + \xi), \omega_{b_j}(iy + \zeta)$  tend nontangentially to  $\xi_j$  and  $\zeta_j$ , respectively, as  $y \downarrow 0$ , and

$$\lim_{y \downarrow 0} \frac{\Im \omega_{a_j}(\xi + iy)}{y} = \frac{\mu_j(\{\xi_j\})}{(\mu_1 \boxplus \mu_2)(\{\xi\})}, \quad \lim_{y \downarrow 0} \frac{\Im \omega_{b_j}(\zeta + iy)}{y} = \frac{\nu_j(\{\zeta_j\})}{(\nu_1 \boxplus \nu_2)(\{\zeta\})}.$$

From the dominated convergence theorem, we know that for any finite measure  $\eta$  in the plane and any sequences  $\{z_n\}_n, \{w_n\}_n \subset \mathbb{C}^+$  which converge nontangentially to  $\xi$  and  $\zeta$ , respectively, we have

$$\lim_{n \rightarrow \infty} (z_n - \xi)(w_n - \zeta)G_\eta(z_n, w_n) = \eta(\{(\xi, \zeta)\}).$$

Applying Proposition 3.5 with  $z = \xi + iy, w = \zeta + iy$  and letting  $y \downarrow 0$  yields

$$\left( \frac{(\mu_1 \boxplus \mu_2)(\{\xi\})(\nu_1 \boxplus \nu_2)(\{\zeta\})}{\mu_j(\{\xi_j\})\nu_j(\{\zeta_j\})} \right)^{\frac{1}{2}} \eta_j(\{(\xi_j, \zeta_j)\}) \geq (\eta_1 \boxplus \eta_2)(\{(\xi, \zeta)\}) > 0.$$

We divide by  $y^2$  in Equation (4) to conclude that

$$\begin{aligned} & \frac{1}{(\eta_1 \boxplus \eta_2)(\{(\xi, \zeta)\})} + \frac{1}{(\mu_1 \boxplus \mu_2)(\{\xi\})(\nu_1 \boxplus \nu_2)(\{\zeta\})} \\ &= \lim_{y \downarrow 0} \frac{1}{y^2 G_{\eta_1 \boxplus \eta_2}(\xi + iy, \zeta + iy)} + \frac{1}{y G_{\mu_1 \boxplus \mu_2}(\xi + iy) y G_{\nu_1 \boxplus \nu_2}(\zeta + iy)} \\ &= \sum_{j=1}^2 \left[ \lim_{y \downarrow 0} \frac{\Im \omega_{a_j}(\xi + iy) \Im \omega_{b_j}(\zeta + iy)}{y^2} \right. \\ & \quad \left. \times \lim_{y \downarrow 0} \frac{1}{\Im \omega_{a_j}(\xi + iy) \Im \omega_{b_j}(\zeta + iy) G_{\eta_j}(\omega_{a_j}(\xi + iy), \omega_{b_j}(\zeta + iy))} \right] \\ &= \frac{1}{(\mu_1 \boxplus \mu_2)(\{\xi\})(\nu_1 \boxplus \nu_2)(\{\zeta\})} \left( \frac{\mu_1(\{\xi_1\})\nu_1(\{\zeta_1\})}{\eta_1(\{(\xi_1, \zeta_1)\})} + \frac{\mu_2(\{\xi_2\})\nu_2(\{\zeta_2\})}{\eta_2(\{(\xi_2, \zeta_2)\})} \right). \end{aligned}$$

□

We conclude this section with a simple remark generalizing the linearization result from [6] to bi-free bi-partite selfadjoint random variables.

**Proposition 3.7.** *Assume that  $(a_1, b_1), (a_2, b_2) \in \mathcal{A}^2$  selfadjoint random variables in the scalar-valued  $*$ -probability space  $(\mathcal{A}, \varphi)$  which are bi-free with respect to  $\varphi$  and satisfy  $a_j b_j = b_j a_j$ ,  $j = 1, 2$ . Consider selfadjoint polynomials  $p, q$  in two noncommuting indeterminates. Then there exist  $m \in \mathbb{N}$ ,  $\alpha_j, \beta_j, \gamma_j \in M_{m+1}(\mathbb{C})$  which are selfadjoint such that*

$$\begin{aligned} & \varphi((z - p(a_1, a_2))^{-1}(w - q(b_1, b_2))^{-1}) = \\ & [G_{(a_1 \otimes \alpha_1 + a_2 \otimes \alpha_2, b_1 \otimes \beta_1 + b_2 \otimes \beta_2)}(ze_{1,1} + \gamma_1, we_{1,1} + \gamma_2)]_{1, m+2}. \end{aligned}$$

Moreover,  $G_{(a_1 \otimes \alpha_1 + a_2 \otimes \alpha_2, b_1 \otimes \beta_1 + b_2 \otimes \beta_2)}$  can be computed via the analytic subordination functions provided by Proposition 3.4.

*Proof.* As shown in [1] (see also [6, Section 3]), for any selfadjoint polynomial in two noncommuting selfadjoint indeterminates  $p \in \mathbb{C}\langle X_1, X_2 \rangle$ , one can find  $m_p \in \mathbb{N}$

and a selfadjoint matrix  $L_p = \begin{bmatrix} 0 & -u_p^* \\ -u_p & -Q_p \end{bmatrix} \in M_{m_p+1}(\mathbb{C}\langle X_1, X_2 \rangle)$  such that

- (1) each entry of  $L_p$  is of degree less than or equal to one,
- (2)  $u_p \in M_{m_p \times 1}(\mathbb{C}\langle X_1, X_2 \rangle)$ ,
- (3)  $Q_p \in M_{m_p}(\mathbb{C}\langle X_1, X_2 \rangle)$  is invertible with  $Q_p^{-1} \in M_{m_p}(\mathbb{C}\langle X_1, X_2 \rangle)$ , and
- (4)  $[(ze_{1,1} - L_p)^{-1}]_{1,1} = \left( \begin{bmatrix} z & u_p^* \\ u_p & Q_p \end{bmatrix}^{-1} \right)_{1,1} = (z - p)^{-1}$ , that is,  $p = u_p^* Q_p^{-1} u_p$ .

Clearly, such an  $L_p$  needs not be unique.

Choose now such a matrix  $L_p \in M_{m_p+1}(\mathbb{C}\langle X_1, X_2 \rangle)$ , and another matrix  $L_q \in M_{m_q+1}(\mathbb{C}\langle X_1, X_2 \rangle)$  satisfying the same properties. We would like to evaluate  $L_p$  in  $X_1 = a_1$ ,  $X_2 = a_2$  and  $L_q$  in  $X_1 = b_1$ ,  $X_2 = b_2$  and apply Lemma 2.1 and Remark 3.2. In order to be able to do that, we need that  $m_p = m_q$ . Unfortunately there is no apriori reason for that to happen. Assume without loss of generality that  $m_p < m_q$ . We show next that we can modify  $L_p$  such that it still satisfies items (1)–(4) above, but with  $m_p$  replaced by  $m_p + r$  for any  $r \in \mathbb{N}$ . Indeed, if  $L_p$  satisfies (1)–(4) above, then

$$\left( ze_{1,1} - \begin{bmatrix} L_p & 0_{m_p \times r} \\ 0_{r \times m_p} & 1_{r \times r} \end{bmatrix} \right)^{-1} = \begin{bmatrix} (ze_{1,1} - L_p)^{-1} & 0_{m_p \times r} \\ 0_{r \times m_p} & -1_{r \times r} \end{bmatrix}.$$

Thus, if our first choice of  $L_p$  and  $L_q$  have different sizes, we complete the smaller one with an identity matrix of the desired size in the lower right corner (and zero elsewhere) in order to make them of equal size  $m + 1$ . Then

$$\begin{aligned} & \begin{bmatrix} z & u_p^* & -1 & 0_{1 \times m} \\ u_p & Q_p & 0_{m \times 1} & 0_{m \times m} \\ 0 & 0_{1 \times m} & w & u_q^* \\ 0_{m \times 1} & 0_{m \times m} & u_q & Q_q \end{bmatrix}^{-1} \\ &= \begin{bmatrix} (z-p)^{-1} & \star & (z-p)^{-1}(w-q)^{-1} & \star \\ \star & \star & \star & \star \\ 0 & 0_{1 \times m} & (w-q)^{-1} & \star \\ 0_{m \times 1} & 0_{m \times m} & \star & \star \end{bmatrix} \end{aligned}$$

We evaluate  $u_p, Q_p$  in  $a_1$  and  $a_2$ ,  $u_q, Q_q$  in  $b_1$  and  $b_2$ , and apply  $\varphi$ . The proposition follows now easily from Lemma 2.1 and Remark 3.2.  $\square$

#### 4. BI-FREE CONVOLUTION SEMIGROUPS

One remarkable feature of free additive convolution is the existence of partial free convolution semigroups: for any Borel probability measure  $\mu$  on  $\mathbb{R}$ , there exists a family  $(\mu_t)_{t \geq 1}$  of Borel probability measures on  $\mathbb{R}$  such that  $\mu_1 = \mu$  and  $\mu_{s+t} = \mu_s \boxplus \mu_t$  for all  $s, t \geq 1$ . This phenomenon was first noted in [8] for  $t$  large enough, and proved in [13] for all  $t \geq 1$ . In [5], an analytic subordination formula for  $\mu_t$  to  $\mu_1$  is provided: for any  $t \geq 1$  and  $z \in \mathbb{C}^+$ , there exists  $\omega_\mu(t, z) \in \mathbb{C}^+$  such that

$$\omega_\mu(t, z) = \frac{z}{t} + \left(1 - \frac{1}{t}\right) \frac{1}{G_\mu(\omega_\mu(t, z))}.$$

Moreover,  $G_\mu(\omega_\mu(t, z)) = G_{\mu_t}(z)$ ,  $z \in \mathbb{C}$ , and the correspondence  $z \mapsto \omega_\mu(t, z)$  is analytic on  $\mathbb{C}^+$ . It is easy to see that the above equation uniquely determines  $\omega_\mu(t, z)$ , and hence  $\mu_t$ . The paper [13] provides also an operatorial construction of  $\mu_t$ : if  $a = a^*$  in some  $*$ -probability space  $(\mathcal{A}, \varphi)$  has distribution  $\mu$  with respect to  $\varphi$  and  $p = p^* = p^2$  is a projection which is free from  $a$  and satisfies  $\varphi(p) = 1/t$ , then the distribution of  $pap$  in the reduced algebra  $(p\mathcal{A}p, \frac{1}{\varphi(p)}\varphi(p \cdot p))$  is  $\mu_t$ . Using this construction, one of us has generalized, together with Huang and Mingo, the result of Nica and Speicher to bi-free additive convolution.

More precisely, it has been shown in [11, Theorem 5.2] that for any compactly supported Borel probability measure  $\eta$  on  $\mathbb{R}^2$ , there exists a partially defined bi-free convolution semigroup  $(\eta_t)_{t \geq 1}$  satisfying the conditions  $\eta_1 = \eta$  and

$\eta_{s+t} = \eta_s \boxplus \eta_t$  for all  $s, t \geq 1$ . As expected, we have  $R_{\eta_t}(z, w) = tR_\eta(z, w)$ ,  $t \geq 1$ . The partial semigroup  $(\eta_t)_{t \geq 1}$  extends to a full weakly continuous semigroup  $[0, +\infty) \ni t \mapsto \eta_t$  with  $\eta_0 = \delta_{(0,0)}$  if and only if  $\eta$  is bi-freely infinitely divisible (see [11, Theorem 4.2]).

Consider a  $*$ -probability space  $(\mathcal{A}, \varphi)$ . For any  $t \geq 1$ , let  $(a_t, b_t) \in \mathcal{A}^2$  be a two-faced pair of noncommutative random variables such that  $a_t b_t = b_t a_t$ ,  $a_t = a_t^*$ , and  $b_t = b_t^*$ , and the distribution of  $(a_t, b_t)$  with respect to  $\varphi$  equals  $\eta_t$ . Denote  $\mu_t$  the distribution of  $a_t$  and  $\nu_t$  the distribution of  $b_t$  with respect to  $\varphi$ . Define  $X_t = \begin{bmatrix} a_t & 0 \\ 0 & b_t \end{bmatrix} \in M_2(\mathcal{A})$ . As seen in Lemma 3.1, we have

$$\begin{aligned} R_{X_t} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) &= \begin{bmatrix} R_{a_t}(z) & z^{-1}\zeta w^{-1}(R_{(a_t, b_t)}(z, w) - zR_{a_t}(z) - wR_{b_t}(w)) \\ 0 & R_{b_t}(w) \end{bmatrix} \\ &= t \begin{bmatrix} R_{a_1}(z) & z^{-1}\zeta w^{-1}(R_{(a_1, b_1)}(z, w) - zR_{a_1}(z) - wR_{b_1}(w)) \\ 0 & R_{b_1}(w) \end{bmatrix} \\ &= tR_X \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right). \end{aligned}$$

As shown in [2, Theorem 7.9], for a given  $X_1 = X_1^* \in M_2(\mathcal{A})$  and  $t \geq 1$ , there exists an  $\tilde{X}_t = \tilde{X}_t^*$  such that  $R_{\tilde{X}_t} = tR_{X_1}$ . By restricting  $R_{\tilde{X}_t}$  to the set of upper triangular matrices and applying Lemma 3.1, we see that  $R_{\tilde{X}_t} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) = R_{X_t} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right)$  for all  $z, w, \zeta \in \mathbb{C}$  of sufficiently small absolute value. In particular, it follows that the subordination formula of [2, Theorem 8.4] holds for  $X_t$ : there exists a function  $\omega_{X_t}$  defined on the set of elements  $\begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix}$  with strictly positive imaginary part which satisfies the functional equation

$$\omega_{X_t} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) = \frac{1}{t} \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} + \left( 1 - \frac{1}{t} \right) G_{X_1} \left( \omega_{X_t} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) \right)^{-1},$$

and  $G_{X_1} \circ \omega_{X_t} = G_{X_t}$ . The point  $\omega_{X_t} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right)$  is the unique attracting fixed point of the map  $v \mapsto \frac{1}{t} \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} + \left( 1 - \frac{1}{t} \right) G_{X_1}(v)^{-1}$ . Since this map sends upper triangular matrices to upper triangular matrices, it follows that  $\omega_{X_t} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) = \begin{bmatrix} f_1(z, \zeta, w) & f_2(z, \zeta, w) \\ 0 & f_3(z, \zeta, w) \end{bmatrix}$  itself is upper triangular. Its entries are easily determined by using the above-displayed equation:

$$\begin{aligned} \begin{bmatrix} f_1(z, \zeta, w) & f_2(z, \zeta, w) \\ 0 & f_3(z, \zeta, w) \end{bmatrix} &= \frac{1}{t} \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \\ &\quad + \left( 1 - \frac{1}{t} \right) \begin{bmatrix} G_{a_1}(f_1(z, \zeta, w))^{-1} & \frac{\varphi((f_1(z, \zeta, w) - a_1)^{-1} f_2(z, \zeta, w) (f_3(z, \zeta, w) - b_1)^{-1})}{G_{a_1}(f_1(z, \zeta, w)) G_{b_1}(f_3(z, \zeta, w))} \\ 0 & G_{b_1}(f_3(z, \zeta, w))^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \frac{z}{t} + \left( 1 - \frac{1}{t} \right) \frac{1}{G_{a_1}(f_1(z, \zeta, w))} & \frac{1}{t} \zeta + \left( 1 - \frac{1}{t} \right) \frac{f_2(z, \zeta, w) G_{(a_1, b_1)}(f_1(z, \zeta, w), f_3(z, \zeta, w))}{G_{a_1}(f_1(z, \zeta, w)) G_{b_1}(f_3(z, \zeta, w))} \\ 0 & \frac{w}{t} + \left( 1 - \frac{1}{t} \right) \frac{1}{G_{b_1}(f_3(z, \zeta, w))} \end{bmatrix}. \end{aligned}$$

The equalities corresponding to entries (1, 1) and (2, 2) provide as indicated at the beginning of this section, via [5, Theorem 2.5], that  $f_1(z, \zeta, w) = \omega_{\mu_1}(t, z)$ ,  $f_3(z, \zeta, w) = \omega_{\nu_1}(t, w)$ . The (1, 2) corner provides the relation

$$\begin{aligned} f_2(z, \zeta, w) &= \zeta \frac{G_{a_1}(\omega_{\mu_1}(t, z))G_{b_1}(\omega_{\nu_1}(t, w))}{tG_{a_1}(\omega_{\mu_1}(t, z))G_{b_1}(\omega_{\nu_1}(t, w)) + (1-t)G_{(a_1, b_1)}(\omega_{\mu_1}(t, z), \omega_{\nu_1}(t, w))} \\ &= \zeta \frac{G_{a_t}(z)G_{b_t}(w)}{tG_{a_t}(z)G_{b_t}(w) + (1-t)G_{(a_1, b_1)}(\omega_{\mu_1}(t, z), \omega_{\nu_1}(t, w))}. \end{aligned}$$

Thus, using  $G_{X_1} \circ \omega_{X_t} = G_{X_t}$  we obtain a formula for the Cauchy transform of a measure in a partial bi-free additive convolution semigroup:

$$(12) \quad G_{(a_t, b_t)}(z, w) = \frac{1}{\frac{t}{G_{(a_1, b_1)}(\omega_{\mu}(t, z), \omega_{\nu}(t, w))} + \frac{1-t}{G_{a_1}(\omega_{\mu}(t, z))G_{b_1}(\omega_{\nu}(t, w))}}, \quad z, w \in \mathbb{C}^+.$$

An analogue of Proposition 3.5 now easily follows:

**Proposition 4.1.** *Let  $(\mathcal{A}, \varphi)$  be a  $*$ -probability space. Assume that for any  $t \geq 1$ , there is a two-faced pair  $(a_t, b_t) \in \mathcal{A}^2$  of noncommutative random variables such that  $a_t b_t = b_t a_t$ ,  $a_t = a_t^*$ ,  $b_t = b_t^*$ , the distribution of  $(a_t, b_t)$  with respect to  $\varphi$  equals  $\eta_t$ , and  $\eta_{s+t} = \eta_s \boxplus \eta_t$ ,  $s, t \geq 1$ . Denote  $\mu_t$  the distribution of  $a_t$  and  $\nu_t$  the distribution of  $b_t$ . Then*

$$\Im \omega_{\mu}(t, z) \Im \omega_{\nu}(t, w) |G_{(a_1, b_1)}(\omega_{\mu}(t, z), \omega_{\nu}(t, w))|^2 \geq \Im z \Im w |G_{(a_t, b_t)}(z, w)|^2,$$

for all  $z, w \in \mathbb{C}^+$ .

*Proof.* The inequality follows from the fact that  $\Im \omega_{X_t} \left( \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} \right) > 0$  whenever

$$\Im \begin{bmatrix} z & \zeta \\ 0 & w \end{bmatrix} > 0 \text{ in } M_2(\mathbb{C}) \text{ and from the relation}$$

$$G_{(a_t, b_t)}(z, w) = f_2(z, \zeta, w) G_{(a_1, b_1)}(\omega_{\mu_1}(t, z), \omega_{\nu_1}(t, w)).$$

The proof is identical to the proof of Proposition 3.5 and is left as an exercise to the reader.  $\square$

**Corollary 4.2.** *Consider a compactly supported Borel probability measure  $\eta$  on  $\mathbb{R}^2$  and let  $t > 1$  be given. Let  $(\eta_t)_{t \geq 1}$  be its partial bi-free convolution semigroup. Assume that there is a point  $(\xi, \zeta) \in \mathbb{R}^2$  so that  $\eta_t(\{(\xi, \zeta)\}) > 0$ . Then  $\eta(\{(\xi/t, \zeta/t)\}) > 0$  and*

$$\eta_t(\{(\xi, \zeta)\}) = \frac{(t\mu(\{\xi/t\}) + 1 - t)(t\nu(\{\zeta/t\}) + 1 - t)\eta(\{(\xi/t, \zeta/t)\})}{t\mu(\{\xi/t\})\nu(\{\zeta/t\}) + (1 - t)\eta(\{(\xi/t, \zeta/t)\})}$$

*Proof.* The presence of an atom of  $\eta_t$  at  $(\xi, \zeta)$  implies the presence of atoms for the marginals  $\mu_t$  (at  $\xi$ ) and  $\nu_t$  (at  $\zeta$ ), respectively. Thus, as shown in [5, Theorem 3.1], we have

$$\begin{aligned} (1) \quad & \lim_{y \downarrow 0} \omega_{\mu}(t, \xi + iy) = \xi/t, \quad \lim_{y \downarrow 0} \omega_{\nu}(t, \zeta + iy) = \zeta/t; \\ (2) \quad & \mu_t(\{\xi\}) = t\mu(\{\xi/t\}) + 1 - t, \quad \nu_t(\{\zeta\}) = t\nu(\{\zeta/t\}) + 1 - t; \\ (3) \quad & \lim_{y \downarrow 0} \frac{\Im \omega_{\mu}(t, \xi + iy)}{y} = \frac{1}{t} + (1 - \frac{1}{t}) \frac{1}{\mu_t(\{\xi\})} = \frac{\mu(\{\xi/t\})}{t\mu(\{\xi/t\}) + 1 - t} \text{ and} \\ & \lim_{y \downarrow 0} \frac{\Im \omega_{\nu}(t, \zeta + iy)}{y} = \frac{1}{t} + (1 - \frac{1}{t}) \frac{1}{\nu_t(\{\zeta\})} = \frac{\nu(\{\zeta/t\})}{t\nu(\{\zeta/t\}) + 1 - t}. \end{aligned}$$



In particular,  $\mu(\{\xi/t\}) > 1 - 1/t$  and  $\nu(\{\zeta/t\}) > 1 - 1/t$ . Applying Proposition 4.1 to  $z = \xi + iy$ ,  $w = \zeta + iy$  and taking limit as  $y \rightarrow 0$  we obtain

$$\left( \frac{(t\mu(\{\xi/t\}) + 1 - t)(t\nu(\{\zeta/t\}) + 1 - t)}{\mu(\{\xi/t\})\nu(\{\zeta/t\})} \right)^{\frac{1}{2}} \eta(\{(\xi/t, \zeta/t)\}) \geq \eta_t(\{(\xi, \zeta)\}) > 0.$$

Thus,  $\eta(\{(\xi/t, \zeta/t)\}) > 0$ . Multiplying by  $y^2$  in (12) evaluated in  $z = \xi + iy$ ,  $w = \zeta + iy$  and taking limits as  $y$  decreases to zero yields

$$\begin{aligned} & \eta_t(\{(\xi, \zeta)\}) \\ &= \lim_{y \downarrow 0} y^2 G_{(a_t, b_t)}(\xi + iy, \zeta + iy) \\ &= \lim_{y \downarrow 0} \frac{1}{\frac{t\Im\omega_\mu(t, \xi + iy)\Im\omega_\nu(t, \zeta + iy)}{y^2\Im\omega_\mu(t, \xi + iy)\Im\omega_\nu(t, \zeta + iy)G_{(a_1, b_1)}(\omega_\mu(t, \xi + iy), \omega_\nu(t, \zeta + iy))} + \frac{1-t}{yG_{a_t}(\xi + iy)yG_{b_t}(\zeta + iy)}} \\ &= \frac{1}{\frac{t\mu(\{\xi/t\})\nu(\{\zeta/t\})}{(t\mu(\{\xi/t\}) + 1 - t)(t\nu(\{\zeta/t\}) + 1 - t)\eta(\{(\xi/t, \zeta/t)\})} + \frac{1-t}{(t\mu(\{\xi/t\}) + 1 - t)(t\nu(\{\zeta/t\}) + 1 - t)}} \\ &= \frac{(t\mu(\{\xi/t\}) + 1 - t)(t\nu(\{\zeta/t\}) + 1 - t)\eta(\{(\xi/t, \zeta/t)\})}{t\mu(\{\xi/t\})\nu(\{\zeta/t\}) + (1 - t)\eta(\{(\xi/t, \zeta/t)\})}, \end{aligned}$$

which concludes our proof.  $\square$

We record a more elegant version of the relation from the above corollary:

$$(13) \quad \frac{\mu_t(\{\xi\})\nu_t(\{\zeta\})}{\eta_t(\{(\xi, \zeta)\})} = t \frac{\mu(\{\xi/t\})\nu(\{\zeta/t\})}{\eta(\{(\xi/t, \zeta/t)\})} + 1 - t.$$

**Example 4.3.** We compute a simple example: let  $\eta = \frac{3}{4}\delta_{(1,1)} + \frac{1}{8}\delta_{(0,0)} + \frac{1}{8}\delta_{(1,0)}$ . Then  $\mu = \frac{1}{8}\delta_0 + \frac{7}{8}\delta_1$ ,  $\nu = \frac{1}{4}\delta_0 + \frac{3}{4}\delta_1$ . The longest an atom can hope to survive is for as long as  $t < 4$ . Indeed,  $\mu_t(\{0\}) = \max\{0, t\mu(\{0\}) + 1 - t\} = \max\{0, 1 - \frac{7}{8}t\}$ ,  $\mu_t(\{t\}) = \max\{0, 1 - \frac{1}{8}t\}$ ,  $\nu_t(\{0\}) = \max\{0, 1 - \frac{3}{4}t\}$ ,  $\nu_t(\{t\}) = \max\{0, 1 - \frac{1}{4}t\}$ . So if  $t < 8/7$ , then

$$\eta_t(\{(0, 0)\}) = \frac{\frac{1}{8}(1 - \frac{7}{8}t)(1 - \frac{3}{4}t)}{\frac{t}{32} + \frac{1-t}{8}} = \left(1 - \frac{7}{8}t\right),$$

if  $t < 4$ , then

$$\eta_t(\{(t, t)\}) = \frac{\frac{3}{4}(1 - \frac{1}{8}t)(1 - \frac{1}{4}t)}{\frac{21}{24}t + \frac{3}{4}(1 - t)} = \left(1 - \frac{t}{4}\right),$$

and if  $t < 3/4$ , then

$$\eta_t(\{(t, 0)\}) = \frac{\frac{1}{8}(1 - \frac{1}{8}t)(1 - \frac{3}{4}t)}{\frac{7}{32}t + \frac{1-t}{8}} = \frac{1}{8} \frac{(8 - t)(4 - 3t)}{4 + 3t}.$$

A direct computation shows that the sum of the mass of the three atoms is strictly less than one for any  $t > 1$ , so that a nonatomic part occurs immediately after  $t = 1$ , as in the case of free convolution of measures on  $\mathbb{R}$ .

Unlike for free convolution semigroups, the expression for the non-atomic part of  $\eta_t$  is much more unwieldy. Indeed, while in principle formula (12) allows for a direct computation of  $G_{\eta_t}$ , the actual computation, even for such a simple measure as the one from Example 4.3, becomes uncomfortably long. We provide here just the necessary ingredients: The reciprocals of the Cauchy transforms of the marginals at  $t = 1$  are  $G_\mu(z)^{-1} = z - \frac{7z}{8z-1}$  and  $G_\nu(w)^{-1} = w - \frac{3w}{4w-1}$ , and

of the reciprocal of the Cauchy transform of  $\eta$  is  $G_\eta(z, w)^{-1} = \frac{8zw(z-1)(w-1)}{8zw-2z-w+1}$ . For given  $t > 1$ , the subordination functions associated to the two marginals are

$$\omega_\mu(t, z) = \frac{8z + 8 - 7t + \sqrt{[8z - 7t + 6 - 2\sqrt{7(t-1)}][8z - 7t + 6 + 2\sqrt{7(t-1)}]}}{16},$$

and

$$\omega_\nu(t, w) = \frac{4w + 4 - 3t + \sqrt{[4w + 2 - 3t - 2\sqrt{3(t-1)}][4w + 2 - 3t + 2\sqrt{3(t-1)}]}}{8}.$$

Replacing in (12) provides the explicit (algebraic) expression for  $G_{\eta_t}$ .

## 5. NO CONDITIONAL EXPECTATIONS OF THE RESOLVENT

The analytic subordination result of Biane is stronger than the result stated in (1): it is shown in [9] that if  $a_1, a_2$  are free selfadjoint random variables in the *tracial*  $^*$ -noncommutative probability space  $(\mathcal{A}, \varphi)$ , then

$$E_{\mathbb{C}[a_j]}[(z - a_1 - a_2)^{-1}] = (\omega_{a_j}(z) - a_j)^{-1}, \quad z \in \mathbb{C}^+,$$

where  $E_{\mathbb{C}[a_j]}$  denotes the unique trace-preserving conditional expectation from the von Neumann algebra generated by  $a_1$  and  $a_2$  onto the von Neumann algebra generated by  $a_j$ . Voiculescu generalized this result to selfadjoint random variables which are free with amalgamation with respect to a trace-preserving conditional expectation. However, in order to prove formula (1) alone, both in its scalar- and operator-valued version, only analytic function theory methods are needed, as shown in [4, 6]. It is remarkable that one can use this same analytic functions machinery to prove Biane's result, at the cost of an amplification to  $3 \times 3$  matrices. Let us give an outline of this procedure below.

Recall that if  $(\mathcal{A}, \varphi)$  is a tracial  $W^*$ -noncommutative probability space and  $B \subset \mathcal{A}$  is a von Neumann subalgebra, then there exists a unique trace-preserving conditional expectation  $E: \mathcal{A} \rightarrow B$ . This expectation is defined via the following relation: for any  $x \in \mathcal{A}$ ,  $E[x]$  is the unique element in  $B$  so that  $\varphi(x\xi^*) = \varphi(E[x]\xi^*)$  for all  $\xi \in B$ . Clearly, given the hypothesis of weak\*-continuity and faithfulness on  $\varphi$ , it is enough to verify the equality  $\varphi(x\xi^*) = \varphi(E[x]\xi^*)$  for all elements  $\xi$  in a subset of  $B$  whose linear span is dense in  $B$ . Thus, in order to prove the relation  $E_{\mathbb{C}[a_j]}[(z - a_1 - a_2)^{-1}] = (\omega_{a_j}(z) - a_j)^{-1}$ , it suffices to show that for any  $v \in \mathbb{C}[a_j]$ ,  $v > 0$ , we have

$$\varphi((z - a_1 - a_2)^{-1}v) = \varphi((\omega_{a_j}(z) - a_j)^{-1}v).$$

In order to do this, we use a linearization trick similar to the one used in [6] (which originates in G. Anderson paper [1]) and Lemma 2.1, with the right variable equal to

$$\text{zero. Consider } A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -v \\ 0 & -v & a_1 \end{bmatrix} \in M_3(\mathbb{C}[a_1]), A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_2 \end{bmatrix} \in M_3(\mathbb{C}[a_2])$$

(recall that  $v$  is an arbitrary positive element in the von Neumann algebra generated by  $a_1$ ). Lemma 2.1 implies that  $A_1$  and  $A_2$  are free with amalgamation over  $M_3(\mathbb{C})$  with respect to  $M_3(\varphi) := \varphi \otimes \text{Id}_{M_3(\mathbb{C})}$ . According to [6, Theorem 2.7], relation (1)

holds for the  $M_3(\mathbb{C})$ -valued Cauchy transforms of  $A_1, A_2$  and  $A_1 + A_2$ . We have

$$(14) \quad \begin{aligned} G_{A_1+A_2} \left( \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \right) &= M_3(\varphi) \left( \left( \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} - A_1 - A_2 \right)^{-1} \right) \\ &= \begin{bmatrix} \varphi(v(z-a_1-a_2)^{-1}v) & 1 & -\varphi(v(z-a_1-a_2)^{-1}) \\ 1 & 0 & 0 \\ -\varphi((z-a_1-a_2)^{-1}v) & 0 & \varphi((z-a_1-a_2)^{-1}) \end{bmatrix}. \end{aligned}$$

If we denote  $F_A(Z) = G_A(Z)^{-1}$ , then

$$F_{A_1+A_2} \left( \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & \Theta & \frac{\varphi(v(z-a_1-a_2)^{-1})}{G_{a_1+a_2}(z)} \\ 0 & \frac{\varphi((z-a_1-a_2)^{-1}v)}{G_{a_1+a_2}(z)} & \frac{1}{G_{a_1+a_2}(z)} \end{bmatrix},$$

where  $\Theta = \varphi(v(z-a_1-a_2)^{-1})G_{a_1+a_2}(z)^{-1}\varphi((z-a_1-a_2)^{-1}v) - \varphi(v(z-a_1-a_2)^{-1}v)$ . Theorem 2.7 of [6] guarantees (through purely function-theoretic arguments) the existence of subordination functions  $\omega_{A_1}$  and  $\omega_{A_2}$  satisfying (1). This relation together with the fact that  $a_2$  is selfadjoint and thus, by functional calculus, can be treated as a real-valued function, imply via a few arithmetic manipulations and a few applications of the identity principle for analytic functions, that

$$\begin{aligned} \omega_{A_1} \left( \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \right) &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & \theta_2(z) & \theta_3(z) \\ 0 & \theta_3(z) & \omega_{a_1}(z) \end{bmatrix}, \\ \omega_{A_2} \left( \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \right) &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & \tau_2(z) & \tau_3(z) \\ 0 & \tau_3(z) & \omega_{a_2}(z) \end{bmatrix}, \end{aligned}$$

for some analytic functions  $\theta_2, \theta_3, \tau_2, \tau_3$  (the functions  $\omega_{a_j}$  are the subordination functions from formula (1) associated to  $a_j, j = 1, 2$ ). When performing the inversion of the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & \theta_2(z) & \theta_3(z) + v \\ 0 & \theta_3(z) + v & \omega_{a_1}(z) - a_1 \end{bmatrix},$$

taking the expectation and equating with the expression above, we obtain that necessarily

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & \theta_2(z) & \theta_3(z) + v \\ 0 & \theta_3(z) + v & \omega_{a_1}(z) - a_1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{v^2}{\omega_{a_1}(z) - a_1} & 1 & \frac{v}{\omega_{a_1}(z) - a_1} \\ 1 & 0 & 0 \\ \frac{v}{\omega_{a_1}(z) - a_1} & 0 & \frac{1}{\omega_{a_1}(z) - a_1} \end{bmatrix}.$$

(We wrote fractions in the above matrix in order to emphasize that our method here does require that we work in a commutative algebra.) Applying  $M_3(\varphi)$  to the above and recalling that  $\varphi$  is a trace yields  $\varphi(v^2(z-a_1-a_2)^{-1}) = \varphi(v^2(\omega_{a_1}(z) - a_1)^{-1})$ . Since this holds for all  $v > 0$  in the von Neumann algebra generated by  $a_1$ , it follows that  $E_{\mathbb{C}[a_1]}[(z-a_1-a_2)^{-1}] = (\omega_{a_1}(z) - a_1)^{-1}$ .

Based on Lemma 3.1 and Remark 3.2, it would be tempting to use the same trick in order to find  $E_{\mathbb{C}[a_j, b_j]}[(z-a_1-a_2)^{-1}(w-b_1-b_2)^{-1}]$  for  $(a_1, b_1)$  and

$(a_2, b_2)$  in  $\mathcal{A}^2$  bi-free with respect to  $\varphi$  and bi-partite (that is,  $a_j b_j = b_j a_j$ ,  $j = 1, 2$ ). Consider the  $6 \times 6$  matrix

$$V = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & v & 0 & 0 & 0 \\ 0 & v & z - a_1 - a_2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & u \\ 0 & 0 & 0 & 0 & u & w - b_1 - b_2 \end{bmatrix},$$

where  $v \in \mathbb{C}[a_1]$ ,  $u \in \mathbb{C}[b_1]$  are both strictly positive. Lemma 2.1 guarantees that

$$(A_1, B_1) = \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -v \\ 0 & -v & a_1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -u \\ 0 & -u & b_1 \end{bmatrix} \right)$$

and

$$(A_2, B_2) = \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_2 \end{bmatrix} \right)$$

are bi-free with amalgamation over  $M_3(\mathbb{C})$ . For simplicity, let  $Z = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z \end{bmatrix}$ ,

$W = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & w \end{bmatrix}$ , and  $e_{3,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . Proposition 3.4 and Remark 3.2 apply to

$X_j = \begin{bmatrix} A_j & 0 \\ 0 & B_j \end{bmatrix}$ ,  $j = 1, 2$ , and the scalar matrix  $\begin{bmatrix} Z & e_{3,3} \\ 0 & W \end{bmatrix}$ . On the other hand, inverting the matrix  $V$ , we obtain on the two  $3 \times 3$  diagonal blocks precisely the formula from (14) and its analogue for  $b_1, b_2, w$ . In the upper right  $3 \times 3$  corner, we obtain the matrix

$$\begin{bmatrix} v(z - a_1 - a_2)^{-1}(w - b_1 - b_2)^{-1}u & 0 & -v(z - a_1 - a_2)^{-1}(w - b_1 - b_2)^{-1} \\ 0 & 0 & 0 \\ (z - a_1 - a_2)^{-1}(w - b_1 - b_2)^{-1}u & 0 & (z - a_1 - a_2)^{-1}(w - b_1 - b_2)^{-1} \end{bmatrix}.$$

If  $\varphi$  were tracial, applying  $\varphi$  on the above and comparing with the corresponding matrix entry from  $G_{X_1} \left( \omega_{X_1} \left( \begin{bmatrix} Z & e_{3,3} \\ 0 & W \end{bmatrix} \right) \right)$  would provide the bi-free analogue of Biane's result. However, it turns out that  $\varphi$  is tracial only in the relatively trivial case in which the two faces are independent. At this moment it is rather unclear to us whether a different approach would provide a satisfactory formula. We emphasize that a formula stating that  $E_{\mathbb{C}[a_j, b_j]} [(z - a_1 - a_2)^{-1}(w - b_1 - b_2)^{-1}]$  is a product of resolvents of  $a_j$  and  $b_j$  would imply traciality for the restriction of  $\varphi$  to expressions of the type  $v(z - a_1 - a_2)^{-1}(w - b_1 - b_2)^{-1}u$  for  $v \in \mathbb{C}[a_j]$ ,  $u \in \mathbb{C}[b_j]$ .

**Theorem 5.1.** *Let  $(a_1, b_1)$  and  $(a_2, b_2)$  be pairs of self-adjoint operators that are bi-free in a  $*$ -noncommutative probability space  $(\mathcal{A}, \varphi)$ . Suppose that  $\varphi|_{\text{alg}(a_1, a_2, b_1, b_2)}$  is tracial and that for each  $k \in \{1, 2\}$  there does not exist  $\alpha_k, \beta_k \in \mathbb{R}$  such that  $(\varphi(a_k^n), \varphi(b_k^n)) = (\alpha_k^n, \beta_k^n)$  for all  $n \in \mathbb{N}$  (i.e. neither pair is scalars in distribution). Then  $\text{alg}(a_1, a_2)$  and  $\text{alg}(b_1, b_2)$  are independent. In particular,  $\tau$  decomposes as the tensor product of tracial states on  $\text{alg}(a_1, a_2)$  and  $\text{alg}(b_1, b_2)$ .*

*Proof.* By the combinatorial theory of bi-free independence (see [10]) it suffices to prove the following: for all  $n \in \mathbb{N}$ , for all non-constant  $\chi : \{1, \dots, n\} \rightarrow \{\ell, r\}$ , and for all  $k \in \{1, 2\}$  we have

$$\kappa_\chi(c_1, \dots, c_n) = 0$$

where  $c_m = a_k$  if  $\chi(m) = \ell$  and  $c_m = b_k$  if  $\chi(m) = r$ . We will only verify the above when  $k = 1$  as the case  $k = 2$  follows by symmetry. We proceed by induction on  $n$ .

As there does not exist  $\alpha, \beta \in \mathbb{R}$  such that  $(\varphi(a_2^n), \varphi(b_2^n)) = (\alpha^n, \beta^n)$  for all  $n \in \mathbb{N}$ , there exists  $n_1, n_2 \in \mathbb{N}$  such that  $\varphi(a_2^{n_1+n_2}) \neq \varphi(a_2^{n_1})\varphi(a_2^{n_2})$  or  $\varphi(b_2^{n_1+n_2}) \neq \varphi(b_2^{n_1})\varphi(b_2^{n_2})$ . We will assume that  $\varphi(a_2^{n_1+n_2}) \neq \varphi(a_2^{n_1})\varphi(a_2^{n_2})$  as the other case will follow by similar arguments.

The case  $n = 1$  is trivial so we begin with the case  $n = 2$ . Here  $(\chi(1), \chi(2)) \in \{(\ell, r), (r, \ell)\}$ . By bi-freeness and traciality, we know that

$$\begin{aligned} \varphi(a_2^{n_1+n_2})\varphi(a_1 b_1) &= \varphi(a_2^{n_1+n_2} a_1 b_1) \\ &= \varphi(a_2^{n_2} a_1 b_1 a_2^{n_1}) \\ &= \varphi(a_2^{n_1+n_2})\varphi(a_1)\varphi(b_1) + \varphi(a_2^{n_1+1})\varphi(a_2^{n_2})\kappa_{(\ell, r)}(a_1, b_1). \end{aligned}$$

Thus, as

$$\kappa_{(\ell, r)}(a_1, b_1) = \varphi(a_1 b_1) - \varphi(a_1)\varphi(b_1)$$

we obtain that

$$\varphi(a_2^{n_1+n_2})\kappa_{\chi_{1,1}}(a_1, b_1) = \varphi(a_2^{n_1+1})\varphi(a_2^{n_2})\kappa_{(\ell, r)}(a_1, b_1).$$

As  $\varphi(a_2^{n_1+n_2}) \neq \varphi(a_2^{n_1})\varphi(a_2^{n_2})$ , this implies  $\kappa_{(\ell, r)}(a_1, b_1) = 0$ . Similarly,

$$\begin{aligned} \varphi(a_2^{n_1+n_2})\varphi(b_1 a_1) &= \varphi(a_2^{n_1+n_2} b_1 a_1) \\ &= \varphi(a_2^{n_2} b_1 a_1 a_2^{n_1}) \\ &= \varphi(a_2^{n_1+n_2})\varphi(a_1)\varphi(b_1) + \varphi(a_2^{n_1+1})\varphi(a_2^{n_2})\kappa_{(r, \ell)}(b_1, a_1). \end{aligned}$$

Thus the same argument implies  $\kappa_{(r, \ell)}(b_1, a_1) = 0$ .

For the inductive step, suppose we have verified the result for  $n - 1$  for some  $n \in \mathbb{N}$ . Let  $\chi : \{1, \dots, n\} \rightarrow \{\ell, r\}$  be non-constant and let where  $c_m = a_1$  if  $\chi(m) = \ell$  and  $c_m = b_1$  if  $\chi(m) = r$ . Let  $m_1 = |\chi^{-1}(\{\ell\})|$  and let  $m_2 = |\chi^{-1}(\{r\})|$ . By bi-freeness and traciality, we know that

$$\begin{aligned} \varphi(a_2^{n_1+n_2})\varphi(c_1 \cdots c_n) &= \varphi(a_2^{n_1+n_2} c_1 \cdots c_n) \\ &= \varphi(a_2^{n_2} c_1 \cdots c_n a_2^{n_1}) \\ &= \varphi(a_2^{n_1+n_2})\varphi(a_1^{m_1})\varphi(b_1^{m_2}) + \varphi(a_2^{n_1+1})\varphi(a_2^{n_2})\kappa_\chi(c_1, \dots, c_n) \end{aligned}$$

(where we have used the induction hypothesis to deduce any cumulant involving  $a_1$  and  $b_1$  of length at most  $n - 1$  is zero). As

$$\varphi(c_1 \cdots c_n) = \varphi(a_1^{m_1})\varphi(b_1^{m_2}) + \kappa_\chi(c_1, \dots, c_n)$$

(where we have used the induction hypothesis to deduce any cumulant involving  $a_1$  and  $b_1$  of length at most  $n - 1$  is zero), we obtain that

$$\varphi(a_2^{n_1+n_2})\kappa_\chi(c_1, \dots, c_n) = \varphi(a_2^{n_1+1})\varphi(a_2^{n_2})\kappa_\chi(c_1, \dots, c_n).$$

As  $\varphi(a_2^{n_1+n_2}) \neq \varphi(a_2^{n_1})\varphi(a_2^{n_2})$ , this implies  $\kappa_\chi(c_1, \dots, c_n) = 0$ . Hence the result follow.  $\square$

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